4210. *Proposed by Van Khea and Leonard Giugiuc.*

Let ABC be a triangle in which the circumcenter lies on the incircle. Furthermore, let $BC = a, CA = b$ and $AB = c$. For which triangles does the expression $\dfrac{a+b+c}{\sqrt[3]{abc}}$ attain its minimum?

We received 6 submissions, of which 5 were correct and one was incomplete. We present the solution by Arkady Alt.

Let I, O, r, R, and S denote the incenter, circumcenter, inradius, circumradius, and semiperimeter of ΔABC, respectively. It is known (Euler's Theorem) that $OI^2 = R^2 - 2Rr$. By assumption, O lies on the incircle of ΔABC, so $OI = r$. Hence, $R^2 - 2Rr = r^2$ if and only if

$$\left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right) - 1 = 0 \iff \frac{R}{r} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2} \iff R = \left(\sqrt{2}+1\right) r,$$

since $R \neq (1-\sqrt{2})r$. It is known (Emmerich's Inequality, pg. 251 in *Recent Advances in Geometric Inequalities* by D. S. Mitrinović, J. Pečarić, V. Volenec) that for any non-acute triangle T, we have $R \geq \left(\sqrt{2}+1\right) r$ with equality if and only if T is a right isosceles triangle. Hence we may now assume that ΔABC is non-obtuse. Then $\cos A \cos B \cos C \geq 0$. Since

$$\cos A \cos B \cos C = \frac{S^2 - (2R+r)^2}{4R^2},$$

we have $S \geq 2R + r$ which by $R = (\sqrt{2}+1)r$ implies

$$S \geq 2(\sqrt{2}+1)r + r = (2\sqrt{2}+3)r \quad \text{or} \quad \frac{S}{r} \geq 2\sqrt{2}+3.$$

Therefore,

$$\begin{aligned}\frac{(a+b+c)^3}{abc} &= \frac{8S^3}{4RrS} = \frac{2S^2}{Rr} = \frac{2S^2}{(\sqrt{2}+1)r^2} \geq \frac{2(2\sqrt{2}+3)^2}{\sqrt{2}+1} \\ &= 2(17+12\sqrt{2})(\sqrt{2}-1) = 14 + 10\sqrt{2}.\end{aligned}$$

Since equality holds in $\frac{S}{r} \geq 2\sqrt{2}+3$ if and only if $\cos A \cos B \cos C = 0$, we conclude that the lower bound $14 + 10\sqrt{2}$ can be attained only for a right angled triangle. Without loss of generality, we may assume that $C = 90°$.

Since $2R = a + b - 2r$, we have

$$a + b = 2(R+r) = 2((\sqrt{2}+1)r + r) = 2\sqrt{2}(\sqrt{2}+1)r. \tag{1}$$

Also,

$$ab = 2Sr = 2(2\sqrt{2}+3)r^2 = 2(\sqrt{2}+1)^2 r^2. \tag{2}$$

From (1) and (2) we obtain $(a-b)^2 = (a+b)^2 - 4ab = 0$. Hence,

$$a = b = \sqrt{2}(\sqrt{2}+1)r \quad \text{and} \quad c = \sqrt{a^2+b^2} = \sqrt{2}a = 2(\sqrt{2}+1)r.$$

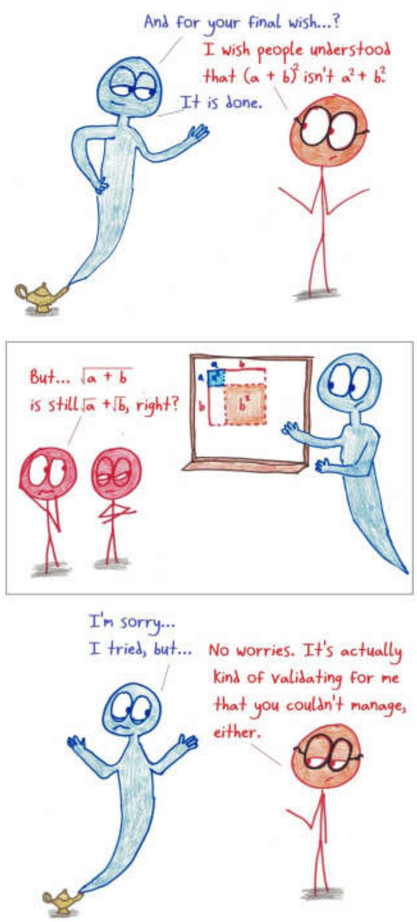
Finally, since

$$\begin{aligned}(\sqrt[3]{2}+\sqrt[6]{32})^3 &= 2+3(2^{2/3})(2^{5/6})+3(2^{1/3})(2^{5/3})+4(2^{1/2})\\ &= 2+3(2^2)+3(2^{3/2})+4(2^{1/2}) = 14+10\sqrt{2},\end{aligned}$$

we have

$$\min\frac{(a+b+c)}{\sqrt[3]{abc}} = \sqrt[3]{14+10\sqrt{2}} = \sqrt[3]{2}+\sqrt[6]{32},$$

attained if and only if ΔABC is a right angled isosceles triangle.



mathwithbaddrawings.com